Inegalitati elementare, metode de baza pentru demonstrarea inegalitatilor

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Metoda reducerii, metoda substitutiei, metoda spargerii, metoda intercalarii

1. $x^2 \geq 0, \forall x \in R$
   Egalitatea are loc <=> x=0

2. Daca $n \in N, n \geq 2$, atunci $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0, (\forall)\, x_i \in \mathbb{R}, i = \overline{1,n}$
   Egalitatea are are loc <=> $x_1 = x_2 = \cdots = x_n = 0$

3. Inegalitatea mediilor:
   a,b>0, atunci $\frac{a+b}{2} \geq \sqrt{a \cdot b} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$
   "=" are loc <=> $a = b$
   Dem:
   I. $\frac{a+b}{2} \geq \sqrt{a \cdot b}$ o demonstram folosind *metoda reducerii*, Aceasta metoda consta in transformarea inegalitatiilor intr-o inegalitate cunoscuta;

      $\frac{a+b}{2} \geq \sqrt{a \cdot b}$ <=> $a + b - 2\sqrt{a \cdot b} \geq 0$ <=> $(\sqrt{a} - \sqrt{b})^2 \geq 0$
      "=" are loc <=> $\sqrt{a} = \sqrt{b}$ <=> $a = b$
   II. $\sqrt{a \cdot b} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$ o demonstram folosind *metoda substitutiei*, aceasta metoda consta in inlocuirea in inegalitati cunoscute a variabilelor cu alte expresii;

      $\sqrt{a \cdot b} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$ <=> $\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{a} \cdot \frac{1}{b}}$
      Ultima inegalitate rezulta din inegalitatea dintre media aritmetica si media geometrica demonstrata anterior ***substituind*** pe a cu $\frac{1}{a}$ si pe b cu $\frac{1}{b}$.
      "=" are loc <=> $\frac{1}{a} = \frac{1}{b}$ <=> $a = b$

Aplicatii:

1) $a+\frac{1}{a}\geq 2, (\forall)\ a>0$

Metoda 1 (metoda reducerii)

$$a+\frac{1}{a}\geq 2 <=> a^2+1\geq 2a <=> a^2-2a+1\geq 0 <=>$$
$$(a-1)^2\geq 0 \text{ (adevarat)}$$

Metoda 2 (metoda substitutiei)

In inegalitatea mediilor

$\frac{a+b}{2}\geq\sqrt{a\cdot b}$ *alegem* $b=\frac{1}{a} => \frac{a+\frac{1}{a}}{2}\geq\sqrt{a\cdot\frac{1}{a}} => a+\frac{1}{a}\geq 2$

1.1) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4, (\forall)\ a,b>0$

1.2) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9, (\forall)\ a,b,c>0$

1.3) $n\in\mathbb{N}, n\geq 2, atunci\ (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\geq n^2, (\forall)a_i>0, i=\overline{1,n}$

Dem:

1.1) *Metoda 1 (metoda reducerii):*

$$2+\frac{a}{b}+\frac{b}{a}\geq 4 <=> a^2+b^2-2ab\geq 0 <=> (a-b)^2\geq 0$$

Metoda 2 (metoda spargerii) → adunarea inegalitatilor → inmultirea inegalitatilor

a) Adunarea inegalitatilor: mai multe inegalitati de acelasi sens se pot aduna obtinandu-se o noua inegalitate

b) Inmultirea inegalitatilor: mai multe inegalitati de acelasi fel si cu toti membrii numere pozitive se pot inmulti obtinandu-se o noua inegalitate.

$$a+b\geq 2\sqrt{a\cdot b}$$
$$\frac{1}{a}+\frac{1}{b}\geq 2\sqrt{\frac{1}{a\cdot b}} \quad *$$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$$

Observatie: in demonstratie “s-a ascuns” si metoda substitutiei si anume in inegalitatea $\frac{1}{a}+\frac{1}{b} \geq 2\sqrt{\frac{1}{a \cdot b}}$

1.2) *Metoda spargerii*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9 <=> 1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1 \geq 9$$

$$<=> \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right) \geq 6$$

$$\frac{a}{b}+\frac{b}{a} \geq 2 \qquad \left(\frac{a}{b}+\frac{1}{\frac{a}{b}} \geq 2\right)$$

$$\frac{a}{c}+\frac{c}{a} \geq 2$$

$$\frac{b}{c}+\frac{c}{b} \geq 2$$

+

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right) \geq 6$$

Observatie: am utilizat metoda subtitutiei si metoda spargerii.

1.3) Tema

4. $a^2+b^2+c^2 \geq ab+bc+ca, (\forall)\ a,b,c \in \mathbb{R}$

Dem:

Metoda 1 (metoda reducerii)

$$a^2+b^2+c^2 \geq ab+bc+ca \,|\cdot 2$$
$$2a^2+2b^2+2c^2-2ab-2bc-2ca \geq 0$$
$$(a^2+b^2-2ab)+(a^2+c^2-2ca)+(b^2+c^2-2bc) \geq 0$$
$$(a-b)^2+(a-c)^2+(b-c)^2 \geq 0, \quad \text{(adevarat)}$$

“=” se realizeaza <=> a=b=c

Metoda 2 (metoda spargerii):

$$a^2+b^2 \geq 2ab$$
$$b^2+c^2 \geq 2bc$$
$$c^2+a^2 \geq 2ca$$

+

$2a^2 + 2b^2 + 2c^2 \geq 2(ab + bc + ca)$

$a^2 + b^2 + c^2 \geq ab + bc + ca$

Aplicatie

1) $a^4 + b^4 + c^4 \geq abc(a + b + c), (\forall)a, b, c \in \mathbb{R}$

Dem:

Metoda substitutiei si *metoda intercalarii*

Metoda intercalarii se bazeaza pe proprietatea de tranzitivitate a relatiei de ordine: $x \geq y$ si $y \geq z \Rightarrow x \geq z$

$a^2 + b^2 + c^2 \geq ab + bc + ca, (\forall)a, b, c \in \mathbb{R}$ (*)

aplicam pentru $a = a^2, b = b^2$ si $c = c^2$ =>

$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$ (1)

Inlocuim in (*) pe a=ab, b=bc, c=ca =>

$a^2b^2 + b^2c^2 + c^2a^2 \geq ab^2c + bc^2a + ca^2b = abc(a + b + c)$ (2)

(1),(2) => $a^4 + b^4 + c^4 \geq abc(a + b + c)$

5. Inegalitatea lui Titu Andreescu

a) $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}, (\forall)a, b \in \mathbb{R}, \; x, y > 0$

b) $\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots \frac{a_n^2}{x_n} \geq \frac{(a_1+a_2+\ldots+a_n)^2}{x_1+x_2+\cdots+x_n}, (\forall)\, a_i \in \mathbb{R}, x_i > 0, i = \overline{1,n}, n \geq 2$

Dem:

a) Metoda reducerii

$a^2y(x + y) + b^2x(x + y) - xy(a + b)^2 \geq 0$

$a^2xy + a^2y^2 + b^2x^2 + b^2xy - a^2xy - 2xyab - xyb^2 \geq 0$

$(ay - bx)^2 \geq 0$ (adevărat)

"=" are loc <=> $ay = bx$, adica $\frac{a}{x} = \frac{b}{y}$ => "=" se realizeaza <=> numerele a si b sunt direct proportionale cu numerele x si y

b) $\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} \geq \frac{(a_1+a_2)^2}{x_1+x_2}$

$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \frac{a_3^2}{x_3} \geq \frac{(a_1+a_2)^2}{x_1+x_2} + \frac{a_3^2}{x_3} \geq \frac{(a_1+a_2+a_3)^2}{x_1+x_2+x_3}$

Se continua rationamentul in n-1 pasi.

Observatie: s-a utilizat metoda substitutiei si metoda intercalarii. O demonstratie a acestei inegalitatii se poate face si prin inductie matematica.

Aplicatie:

1. $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c, (\forall)\, a, b, c > 0$

2. $\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq a+b+c, (\forall)a,b,c>0$
3. $\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+d^2}{c+d}+\frac{d^2+a^2}{d+a}\geq a+b+c+d, (\forall)a,b,c,d>0$

Dem: 1). $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)^2}{a+b+c}=a+b+c$

2.) $\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\right)+\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\right)\geq\frac{(a+b+c)^2}{2(a+b+c)}+\frac{(a+b+c)^2}{2(a+b+c)}=a+b+c$

S-a utilizat metoda spargerii si metoda substitutiei.

3.) Tema

6. Inegaliatea Cauchy-Buniakovski-Schwarz
   a) $(ax+by)^2\leq(a^2+b^2)(x^2+y^2), (\forall)a,b,x,y\in\mathbb{R}$
   b) $(ax+by+cz)^2\leq(a^2+b^2+c^2)(x^2+y^2+z^2), (\forall)a,b,c,x,y,z\in\mathbb{R}$
   c) $(a_1b_1+a_2b_2+\dots+a_nb_n)^2\leq(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2),\ (\forall)\ a_i,b_i\in\mathbb{R}, i=\overline{1,n}, n\geq 2$

Dem:

a) Metoda reducerii

$a^2x^2+2axby+b^2y^2-a^2x^2-a^2y^2-b^2x^2-b^2y^2\leq 0$
$a^2y^2-2abxy+b^2x^2\geq 0$
$(ay-bx)^2\geq 0$ (adevarat)

"=" are loc <=> ay=bx, adica numerele a si b sunt direct proportionale cu numerele x si y

b) Se demonsteaza asemanator cu a)

c) Se demonstreaza folosind identitatea lui Lagrange.

Aplicatie:

1. $(a+b)^2\leq 2(a^2+b^2), (\forall)a,b\in\mathbb{R}$

   Dem:

   In inegalitatea C.B.S luam x=1 si y=1 => $(a+b)^2\leq 2(a^2+b^2)$
2. $(a+b+c)^2\leq 3(a^2+b^2+c^2), (\forall)a,b,c\in\mathbb{R}$
3. $(a_1+a_2+\dots+a_n)^2\leq n(a_1^2+a_2^2+\dots+a_n^2), (\forall)a_i\in\mathbb{R}, i=\overline{1,n}$

   Observatie: inegalitatea 3 mai poate fi scrisa sub forma:

$$\sqrt{\frac{a_1^2+a_2^2+\dots+a_n^2}{n}}\geq\frac{a_1+a_2+\dots+a_n}{n}$$

Media patratica

4. $\frac{a^3+b^3+c^3}{a^2+b^2+c^2} \geq \frac{a^2+b^2+c^2}{a+b+c} \geq \frac{a+b+c}{3}, (\forall) a,b,c > 0$

   Dem:

   - $\frac{a^3+b^3+c^3}{a^2+b^2+c^2} \geq \frac{a^2+b^2+c^2}{a+b+c}$ <=> $(a^3+b^3+c^3)(a+b+c) \geq (a^2+b^2+c^2)^2$

     Aceasta inegaliate se demonstreaza folosind metoda substitutiei in inegalitatea C.B.S de la punctul b) alegand a=$\sqrt{a}$, $x = a\sqrt{a}$

     b=$\sqrt{b}$ $y = b\sqrt{b}$

     c=$\sqrt{c}$ $z = c\sqrt{c}$

     => $(a^2+b^2+c^2)^2 \leq (a+b+c)(a^3+b^3+c^3)$
   - $3(a^2+b^2+c^2) \geq (a+b+c)^2$ (adevarat)

7. Inegalitatea lui Minkovski

a) $\sqrt{(a+b)^2+(c+d)^2} \leq \sqrt{a^2+c^2} + \sqrt{b^2+d^2}, (\forall) a,b,c,d \in \mathbb{R}$

b)

$\sqrt{(a_1+b_1)^2+(a_2+b_2)^2+\cdots+(a_n+b_n)^2} \leq$

$\sqrt{a_1^2+a_2^2+\cdots+a_n^2} + \sqrt{b_1^2+b_2^2+\cdots+b_n^2}, (\forall) a_i, b_i \in \mathbb{R}, i = \overline{1,n}, n \geq 2$

Dem:

a) $(a+b)^2+(c+d)^2 \leq a^2+c^2+b^2+d^2+2\sqrt{(a^2+c^2)(b^2+d^2)}$ <=>

<=> $ab+cd \leq \sqrt{(a^2+c^2)(b^2+d^2)}$ (adevarat)(inegalitatea lui C.B.S)

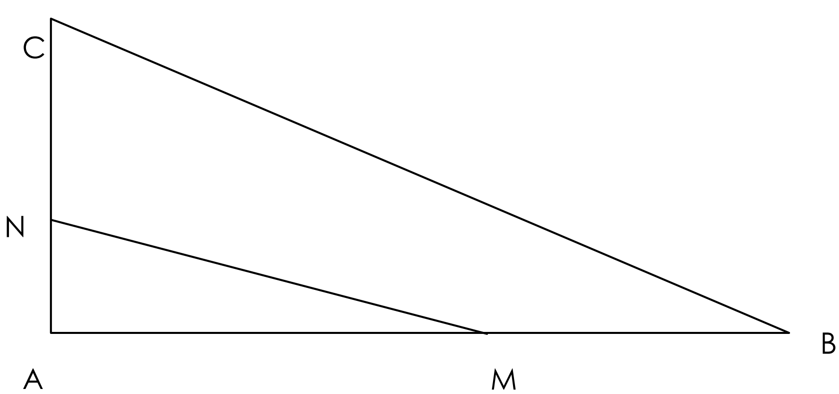
8. Doua probleme de geometrie rezolvate cu ajutorul inegalitatilor algebrice

1) Pe catetele AB si AC ale triunghiului dreptunghic ABC se considera punctele M si N. Sa se demonstreze inegalitatea BC·MN≥AB·AM+AC·AN (1)

2) Fie ABCD patrat si M un punct situat in interiorul patratului. Sa se demonstreze ca punctul M se afla pe una din diagonalele patratului <=> MA·MC+MD·MB=AB^2

   Dem:

1)



Notam AM=x, AN=y, AC=b, AB=c, BC=a

BC=$\sqrt{b^2+c^2}$

$MN=\sqrt{x^2+y^2}$

(1) <=> $\sqrt{(b^2+c^2)(x^2+y^2)} \geq c \cdot x + b \cdot y \quad \Big|^{2}$

$(b^2+c^2)(x^2+y^2) \geq (c \cdot x + b \cdot y)^2$ (adevarat)(inegalitatea lui C. B. S)

2) tema