Techniques of Trigonometric Integration

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

Spring 2011

Outline

Today's discussion will focus on evaluating integrals of the forms:

- $\displaystyle\int \sin^m x \cos^n x \, dx$
- $\displaystyle\int \tan^m x \sec^n x \, dx$
- $\displaystyle\int \cot^m x \csc^n x \, dx$

where $m = 0, 1, \ldots$ and $n = 0, 1, \ldots$.

Powers of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx$$

Powers of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx$$

If m is odd, then $m - 1$ is even, *i.e.* $m - 1 = 2k$ and

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^{m-1} x \cos^n x \sin x \, dx \\
&= \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx \\
&= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx \\
&= -\int \left(1 - u^2\right)^k u^n \, du
\end{aligned}$$

where $u = \cos x$ and $du = -\sin x \, dx$.

$$\int \sin^m x \cos^n x \, dx$$

If n is odd, then $n - 1$ is even, *i.e.* $n - 1 = 2j$ and

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^m x \cos^{n-1} x \cos x \, dx \\
&= \int \sin^m x \left(\cos^2 x\right)^j \cos x \, dx \\
&= \int \sin^m x \left(1 - \sin^2 x\right)^j \cos x \, dx \\
&= \int u^m \left(1 - u^2\right)^j \, du
\end{aligned}$$

where $u = \sin x$ and $du = \cos x \, dx$.

Examples

Example

Evaluate the following indefinite integrals.

1. $\int \sin^3 x \, dx$
2. $\int \sin^2 x \cos^3 x \, dx$

Even Powers of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx$$

If both *m* and *n* are even then we make use of a **half-angle identity**.

$$\begin{aligned}
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x)
\end{aligned}$$

Examples

Example

Evaluate the following indefinite integrals.

1. $\int \sin^2 x \, dx$
2. $\int \cos^4 x \, dx$
3. $\int \sin^2 x \cos^2 x \, dx$

Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$

Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$

If m is odd then $m - 1$ is even, *i.e.*, $m - 1 = 2k$.

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (u^2 - 1)^k u^{n-1} \, du
\end{aligned}$$

where $u = \sec x$ and $du = \sec x \tan x \, dx$.

Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$

If n is even then $n - 2$ is also even, *i.e.*, $n - 2 = 2j$.

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^m x \sec^{n-2} x \sec^2 x \, dx \\
&= \int \tan^m x \left(\sec^2 x\right)^j \sec^2 x \, dx \\
&= \int \tan^m x \left(1 + \tan^2 x\right)^j \sec^2 x \, dx \\
&= \int u^m (1 + u^2)^j \, du
\end{aligned}$$

where $u = \tan x$ and $du = \sec^2 x \, dx$.

Examples

Example

Evaluate the following indefinite integrals.

1. $\int \tan^3 x \sec^3 x \, dx$
2. $\int \tan^2 x \sec^4 x \, dx$

Challenging Case

$$\int \tan^m x \sec^n x \, dx$$

If m is even and n is odd, replace $\tan^2 x$ by $\sec^2 x - 1$. Integration by parts may be necessary.

Challenging Case

$$\int \tan^m x \sec^n x \, dx$$

If m is even and n is odd, replace $\tan^2 x$ by $\sec^2 x - 1$. Integration by parts may be necessary.

Example

Evaluate the following indefinite integral.

$$\int \sec^3 x \, dx$$

Powers of Cotangent and Cosecant

$$\int \cot^m x \csc^n x \, dx$$

Powers of Cotangent and Cosecant

$$\int \cot^m x \csc^n x \, dx$$

If m is odd then $m - 1$ is even, *i.e.*, $m - 1 = 2k$.

$$\begin{aligned}
\int \cot^m x \csc^n x \, dx &= \int \cot^{m-1} x \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\cot^2 x\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&= -\int (u^2 - 1)^k u^{n-1} \, du
\end{aligned}$$

where $u = \csc x$ and $du = -\csc x \cot x \, dx$.

Powers of Cotangent and Cosecant

$$\int \cot^m x \csc^n x \, dx$$

If n is even then $n - 2$ is also even, *i.e.*, $n - 2 = 2j$.

$$\begin{aligned}
\int \cot^m x \csc^n x \, dx &= \int \cot^m x \csc^{n-2} x \csc^2 x \, dx \\
&= \int \cot^m x \left(\csc^2 x\right)^j \csc^2 x \, dx \\
&= \int \cot^m x \left(1 + \cot^2 x\right)^j \csc^2 x \, dx \\
&= -\int u^m (1 + u^2)^j \, du
\end{aligned}$$

where $u = \cot x$ and $du = -\csc^2 x \, dx$.

Examples

Example

Evaluate the following indefinite integrals.

1. $\displaystyle\int \cot^5 x \csc^3 x \, dx$
2. $\displaystyle\int \cot^4 x \csc^4 x \, dx$

Challenging Case

$$\int \cot^m x \csc^n x \, dx$$

If m is even and n is odd, replace $\cot^2 x$ by $\csc^2 x - 1$. Integration by parts may be necessary.

Challenging Case

$$\int \cot^m x \csc^n x \, dx$$

If m is even and n is odd, replace $\cot^2 x$ by $\csc^2 x - 1$. Integration by parts may be necessary.

Example

Evaluate the following indefinite integral.

$$\int \cot^2 x \csc x \, dx$$

Trigonometric Substitution

Remark: We often see expressions of the forms

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2}$$

inside integrands.

In each case there is a trigonometric function which we can substitute for x which may enable us to evaluate the integral.

Case: $\sqrt{a^2 - x^2}$

Suppose the integrand contains an expression of the form $\sqrt{a^2 - x^2}$ with $a > 0$, then we can substitute $x = a \sin \theta$.

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} \\
&= \sqrt{a^2 - a^2 \sin^2 \theta} \\
&= \sqrt{a^2 (1 - \sin^2 \theta)} \\
&= \sqrt{a^2 \cos^2 \theta} \\
&= a \cos \theta
\end{aligned}$$

if $-\pi/2 \leq \theta \leq \pi/2$. (Why?)

Example

Example

Evaluate the following indefinite integral.

$$\int \frac{\sqrt{4 - x^2}}{x^2} \, dx$$

Suppose the integrand contains an expression of the form $\sqrt{a^2 + x^2}$ with $a > 0$, then we can substitute $x = a \tan\theta$.

$$\begin{aligned} \sqrt{a^2 + x^2} &= \sqrt{a^2 + (a\tan\theta)^2} \\ &= \sqrt{a^2 + a^2\tan^2\theta} \\ &= \sqrt{a^2(1 + \tan^2\theta)} \\ &= \sqrt{a^2\sec^2\theta} \\ &= a\sec\theta \end{aligned}$$

if $-\pi/2 < \theta < \pi/2$. (Why?)

Example

Example

Evaluate the following indefinite integral.

$$\int \frac{1}{\sqrt{9+x^2}}\,dx$$

Case: $\sqrt{x^2 - a^2}$

Suppose the integrand contains an expression of the form $\sqrt{x^2 - a^2}$ with $a > 0$, then we can substitute $x = a\sec\theta$.

$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{(a\sec\theta)^2 - a^2} \\ &= \sqrt{a^2\sec^2\theta - a^2} \\ &= \sqrt{a^2(\sec^2\theta - 1)} \\ &= \sqrt{a^2\tan^2\theta} \\ &= a\tan\theta \end{aligned}$$

if $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$. (Why?)

Example

Example

Evaluate the following indefinite integral.

$$\int \frac{\sqrt{x^2 - 4}}{x}\, dx$$

Putting Them All Together

Example

Evaluate the following indefinite integrals.

1. $$\int \frac{1}{x\sqrt{x^2+16}}\,dx$$
2. $$\int \frac{x^2}{\sqrt{9-x^2}}\,dx$$
3. $$\int \frac{1}{(x^2-9)^{3/2}}\,dx$$

Homework

- Read Section 6.3.
- Exercises 1–15 odd and 33 (powers of trigonometric functions)
- Exercises 17–29 odd (trigonometric substitution)