Today’s discussion will focus on evaluating integrals of the forms:

\[ \int \sin^m x \cos^n x \, dx \]
\[ \int \tan^m x \sec^n x \, dx \]
\[ \int \cot^m x \csc^n x \, dx \]

where \( m = 0, 1, \ldots \) and \( n = 0, 1, \ldots \).
Powers of Sine and Cosine

\[
\int \sin^m x \cos^n x \, dx
\]
Powers of Sine and Cosine

\[ \int \sin^m x \cos^n x \, dx \]

If \( m \) is odd, then \( m - 1 \) is even, \( i.e. \) \( m - 1 = 2k \) and

\[ \int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cos^n x \sin x \, dx \]

\[ = \int (\sin^2 x)^k \cos^n x \sin x \, dx \]

\[ = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx \]

\[ = -\int (1 - u^2)^k u^n \, du \]

where \( u = \cos x \) and \( du = -\sin x \, dx \).
Powers of Sine and Cosine

\[ \int \sin^m x \cos^n x \, dx \]

If \( n \) is odd, then \( n - 1 \) is even, i.e. \( n - 1 = 2j \) and

\[
\int \sin^m x \cos^n x \, dx \quad = \quad \int \sin^m x \cos^{n-1} x \cos x \, dx
\]

\[
= \int \sin^m x \left( \cos^2 x \right)^j \cos x \, dx
\]

\[
= \int \sin^m x \left( 1 - \sin^2 x \right)^j \cos x \, dx
\]

\[
= \int u^m \left( 1 - u^2 \right)^j \, du
\]

where \( u = \sin x \) and \( du = \cos x \, dx \).
Examples

Evaluate the following indefinite integrals.

1. \[ \int \sin^3 x \, dx \]
2. \[ \int \sin^2 x \cos^3 x \, dx \]
Even Powers of Sine and Cosine

\[ \int \sin^m x \cos^n x \, dx \]

If both \( m \) and \( n \) are even then we make use of a half-angle identity.

\[
\sin^2 x = \frac{1}{2} (1 - \cos 2x) \\
\cos^2 x = \frac{1}{2} (1 + \cos 2x)
\]
Evaluate the following indefinite integrals.

1. \[ \int \sin^2 x \, dx \]
2. \[ \int \cos^4 x \, dx \]
3. \[ \int \sin^2 x \cos^2 x \, dx \]
Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$
Powers of Tangent and Secant

\[
\int \tan^m x \sec^n x \, dx
\]

If \( m \) is odd then \( m - 1 \) is even, \( i.e., m - 1 = 2k \).

\[
\int \tan^m x \sec^n x \, dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \\
= \int \left( \tan^2 x \right)^k \sec^{n-1} x \sec x \tan x \, dx \\
= \int \left( \sec^2 x - 1 \right)^k \sec^{n-1} x \sec x \tan x \, dx \\
= \int (u^2 - 1)^k u^{n-1} \, du
\]

where \( u = \sec x \) and \( du = \sec x \tan x \, dx \).
Powers of Tangent and Secant

\[ \int \tan^m x \sec^n x \, dx \]

If \( n \) is even then \( n - 2 \) is also even, i.e., \( n - 2 = 2j \).

\[ \int \tan^m x \sec^n x \, dx = \int \tan^m x \sec^{n-2} x \sec^2 x \, dx \]

\[ = \int \tan^m x (\sec^2 x)^j \sec^2 x \, dx \]

\[ = \int \tan^m x (1 + \tan^2 x)^j \sec^2 x \, dx \]

\[ = \int u^m (1 + u^2)^j \, du \]

where \( u = \tan x \) and \( du = \sec^2 x \, dx \).
Example

Evaluate the following indefinite integrals.

1. \[ \int \tan^3 x \sec^3 x \, dx \]
2. \[ \int \tan^2 x \sec^4 x \, dx \]
Challenging Case

\[ \int \tan^m x \sec^n x \, dx \]

If \( m \) is even and \( n \) is odd, replace \( \tan^2 x \) by \( \sec^2 x - 1 \). Integration by parts may be necessary.
\[ \int \tan^m x \sec^n x \, dx \]

If \( m \) is even and \( n \) is odd, replace \( \tan^2 x \) by \( \sec^2 x - 1 \). Integration by parts may be necessary.

**Example**

Evaluate the following indefinite integral.

\[ \int \sec^3 x \, dx \]
Powers of Cotangent and Cosecant

\[ \int \cot^m x \csc^n x \, dx \]
Powers of Cotangent and Cosecant

\[ \int \cot^m x \csc^n x \, dx \]

If \( m \) is odd then \( m - 1 \) is even, \( i.e., m - 1 = 2k \).

\[ \int \cot^m x \csc^n x \, dx = \int \cot^{m-1} x \csc^{n-1} x \csc x \cot x \, dx \]
\[ = \int \left( \cot^2 x \right)^k \csc^{n-1} x \csc x \cot x \, dx \]
\[ = \int \left( \csc^2 x - 1 \right)^k \csc^{n-1} x \csc x \cot x \, dx \]
\[ = -\int (u^2 - 1)^k u^{n-1} \, du \]

where \( u = \csc x \) and \( du = -\csc x \cot x \, dx \).
\[
\int \cot^m x \csc^n x \, dx
\]

If \( n \) is even then \( n - 2 \) is also even, \( i.e., \, n - 2 = 2j. \)

\[
\int \cot^m x \csc^n x \, dx = \int \cot^m x \csc^{n-2} x \csc^2 x \, dx
\]
\[
= \int \cot^m x \left( \csc^2 x \right)^j \csc^2 x \, dx
\]
\[
= \int \cot^m x \left( 1 + \cot^2 x \right)^j \csc^2 x \, dx
\]
\[
= - \int u^m(1 + u^2)^j \, du
\]

where \( u = \cot x \) and \( du = - \csc^2 x \, dx. \)
Evaluate the following indefinite integrals.

1. \[ \int \cot^5 x \csc^3 x \, dx \]
2. \[ \int \cot^4 x \csc^4 x \, dx \]
\[ \int \cot^m x \csc^n x \, dx \]

If \( m \) is even and \( n \) is odd, replace \( \cot^2 x \) by \( \csc^2 x - 1 \). Integration by parts may be necessary.
Challenging Case

\[ \int \cot^m x \csc^n x \, dx \]

If \( m \) is even and \( n \) is odd, replace \( \cot^2 x \) by \( \csc^2 x - 1 \).
Integration by parts may be necessary.

Example

Evaluate the following indefinite integral.

\[ \int \cot^2 x \csc x \, dx \]
Remark: We often see expressions of the forms
\[ \sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2} \]
inside integrands.

In each case there is a trigonometric function which we can substitute for \( x \) which may enable us to evaluate the integral.
Case: $\sqrt{a^2 - x^2}$

Suppose the integrand contains an expression of the form $\sqrt{a^2 - x^2}$ with $a > 0$, then we can substitute $x = a \sin \theta$.

\[
\begin{align*}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} \\
&= \sqrt{a^2 - a^2 \sin^2 \theta} \\
&= \sqrt{a^2 (1 - \sin^2 \theta)} \\
&= \sqrt{a^2 \cos^2 \theta} \\
&= a \cos \theta
\end{align*}
\]

if $-\pi/2 \leq \theta \leq \pi/2$. (Why?)
Example

Evaluate the following indefinite integral.

\[ \int \frac{\sqrt{4 - x^2}}{x^2} \, dx \]
Case: $\sqrt{a^2 + x^2}$

Suppose the integrand contains an expression of the form $\sqrt{a^2 + x^2}$ with $a > 0$, then we can substitute $x = a \tan \theta$.

\[
\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan \theta)^2} \\
= \sqrt{a^2 + a^2 \tan^2 \theta} \\
= \sqrt{a^2 (1 + \tan^2 \theta)} \\
= \sqrt{a^2 \sec^2 \theta} \\
= a \sec \theta
\]

if $-\pi/2 < \theta < \pi/2$. (Why?)
Example

Evaluate the following indefinite integral.

\[\int \frac{1}{\sqrt{9 + x^2}} \, dx\]
Case: $\sqrt{x^2 - a^2}$

Suppose the integrand contains an expression of the form $\sqrt{x^2 - a^2}$ with $a > 0$, then we can substitute $x = a \sec \theta$.

\[
\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 (\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a \tan \theta
\]

if $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$. (Why?)
Example

Evaluate the following indefinite integral.

\[ \int \frac{\sqrt{x^2 - 4}}{x} \, dx \]
Example

Evaluate the following indefinite integrals.

1. \( \int \frac{1}{x\sqrt{x^2 + 16}} \, dx \)
2. \( \int \frac{x^2}{\sqrt{9 - x^2}} \, dx \)
3. \( \int \frac{1}{(x^2 - 9)^{3/2}} \, dx \)
Homework

- Read Section 6.3.
- Exercises 1–15 odd and 33 (powers of trigonometric functions)
- Exercises 17–29 odd (trigonometric substitution)