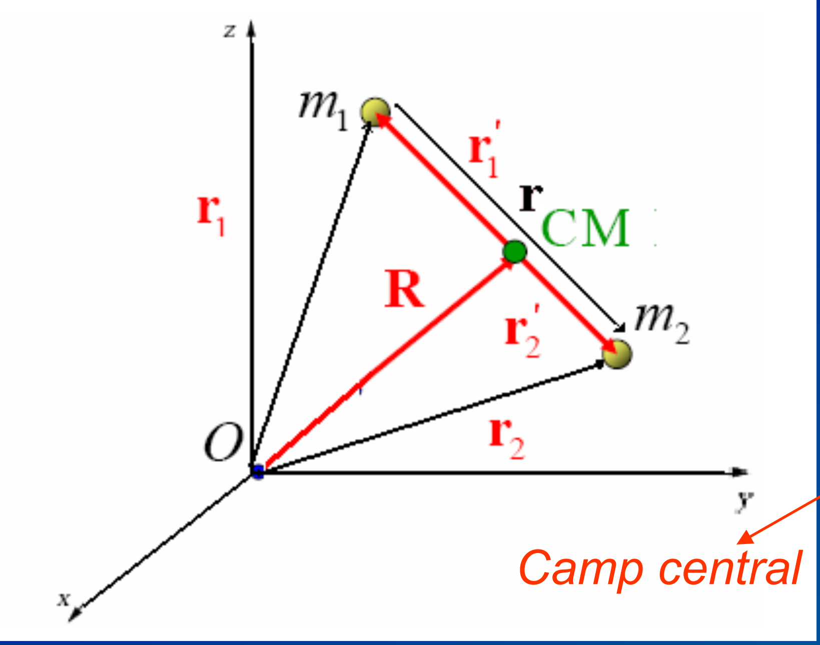
Miacarea in Camp Central de Forte

Problema celor doua corpuri



Ne propunem determinarea ecuatiilor de miscare pentru doua corpuri de mase m_1 si m_2 care interactioneaza unul cu altul prin intermediul unui camp central de forta.

$$\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}} = -\frac{\partial U}{\partial r}\hat{\mathbf{r}}; \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|}; \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Informatii mai bune putem obtine exprimand pozitia centrului de masa (CM)

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Dorim deci , exprimarea vectorilor $\mathbf{r}_1$ *si* $\mathbf{r}_2$ *ca functii de* $\mathbf{r}$ *si* $\mathbf{R}$

$$\left.\begin{aligned} \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ \frac{(m_1 + m_2)\mathbf{R}}{m_2} &= \frac{m_1}{m_2}\mathbf{r}_1 + \mathbf{r}_2 \end{aligned}\right\}(+) \quad \Rightarrow \quad \mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\left(\mathbf{r} + \frac{(m_1 + m_2)\mathbf{R}}{m_2}\right)$$

$$r_1 = R + \frac{m_2}{m_1 + m_2} r \qquad r_2 = R - \frac{m_1}{m_1 + m_2} r$$

Functia Lagrange pentru sistem devine:

$$L = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(r) =$$

$$= \frac{m_1}{2}\left(\dot{R} + \frac{m_2}{M}\dot{r}\right)^2 + \frac{m_2}{2}\left(\dot{R} - \frac{m_1}{M}\dot{r}\right)^2 - U(r); \quad M = m_1 + m_2$$

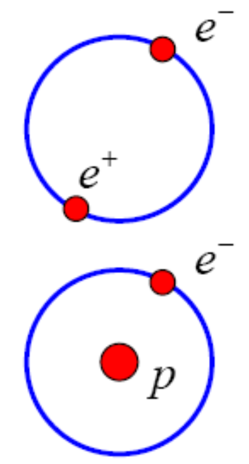
notand $$\mu = \frac{m_1 m_2}{m_1 + m_2}$$ ***masa redusa a sistemului***

$$L = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(r)$$

$$\mu_{\text{positronium}} = \frac{m_e m_e}{(m_e + m_e)} = \frac{m_e}{2}$$

$$m(p) >> m(e^+)$$

$$\mu_{\text{hydrogen}} = \frac{m_p m_e}{(m_p + m_e)} \approx m_e$$



$$V(r) = -\frac{q^2}{r}$$

Aceleasi rezultatele obtinem alegand CM ca noua origine a sistemului

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R}$$

$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R}$$

Teorema lui König

$$m_1\mathbf{r}_1' + m_2\mathbf{r}_2' = 0$$

notand

$$M = m_1 + m_2$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2'$$

$$\mathbf{r}_1' = \frac{m_2}{M}\mathbf{r} \qquad \mathbf{r}_2' = -\frac{m_1}{M}\mathbf{r}$$

$$L = \frac{1}{2}M\mathbf{V}_{CM}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)$$

a)

$$\frac{\partial L}{\partial \dot{\mathbf{R}}} = M\dot{\mathbf{R}}$$

$$\frac{\partial L}{\partial \mathbf{R}} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}}\right) - \frac{\partial L}{\partial \mathbf{R}} = 0 \quad \Rightarrow \quad M\ddot{\mathbf{R}} = 0 \quad \Rightarrow \quad M\dot{\mathbf{R}} = const.$$

$$\begin{cases} M\dot{\mathbf{R}} = M\left(\dfrac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1+m_2}\right) = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \mathbf{p}_1 + \mathbf{p}_2 \\ M\dot{\mathbf{R}} = \mathbf{p}_{Total} \end{cases}$$

$$\Rightarrow \quad \mathbf{p}_{Total} = \mathbf{p}_1 + \mathbf{p}_2$$

Impulsul total se conserva

b)

$$\frac{\partial L}{\partial \dot{\mathbf{r}}} = \mu \dot{\mathbf{r}}$$

$$\frac{\partial L}{\partial \mathbf{r}} = -\frac{\partial U}{\partial r}\frac{\mathbf{r}}{r} = -\frac{\partial U}{\partial r}\hat{\mathbf{r}}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \Rightarrow \quad \mu \ddot{\mathbf{r}} + \frac{\partial U}{\partial r}\hat{\mathbf{r}} = 0$$

⇒ ***Miscarea a doua puncte materiale care interactioneaza intre ele , se reduce la problema miscarii unui punct de masa μ intr-un camp exterior***

Observam ca a) si b) nu sunt cuplate si deci miscarea CM R(t) este decuplata de miscarea relativa r(t)

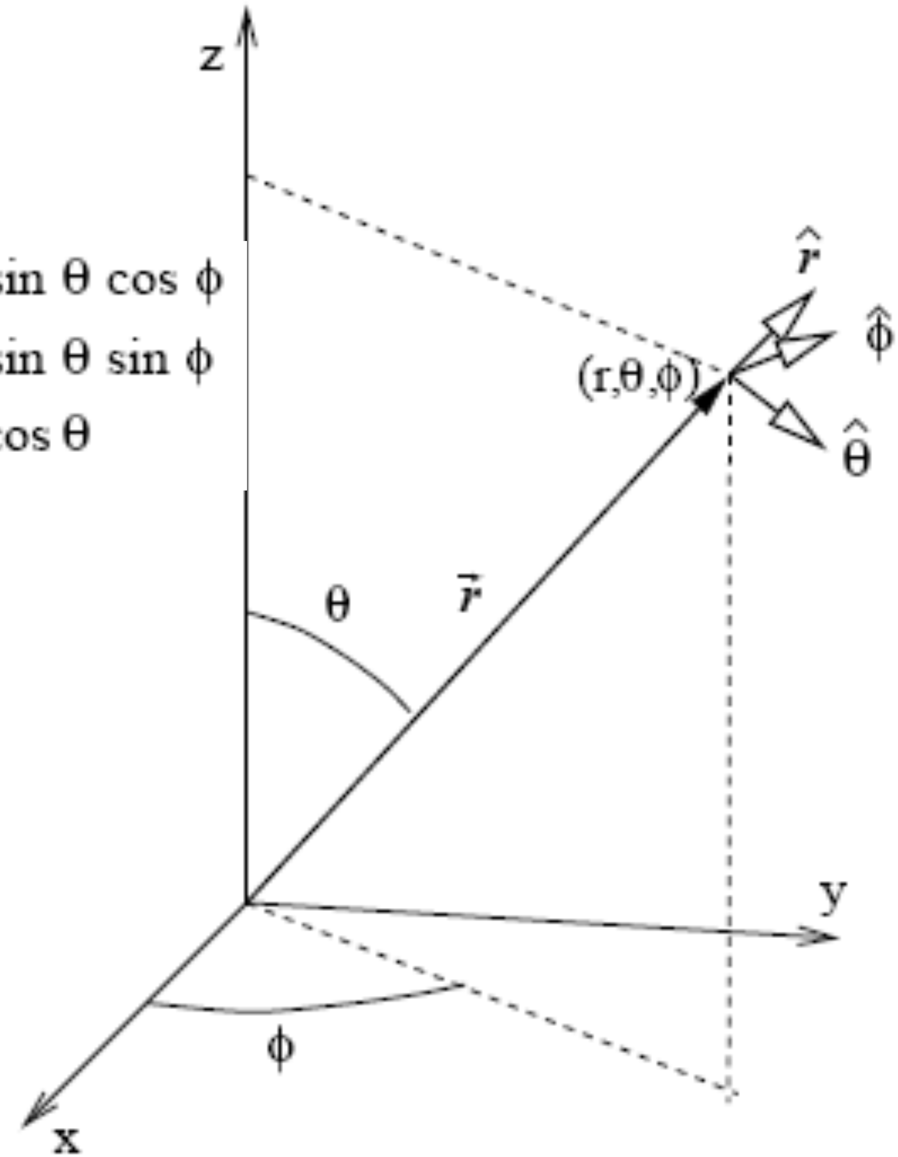
⇒ ***Putem ignora miscarea CM (R(t))***

Sistemul campului central de forta are simetrie sferica

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$



Se poare roti in jurul oricarei axe ce trece prin origine

Simetrie rotationala

- *Lagrangianul nu depinde de directie*

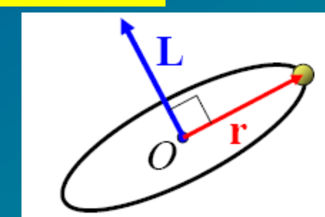
$$L = T(\dot{r}^2) - U(r)$$

Momentul unghiular se conserva

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = const.$$

$$\mathbf{L} \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{r} = 0$$

$$\mathbf{r} \perp \mathbf{L}$$



Traiectoria r(t) este continuta in intregime intr-un plan ortogonal cu L

Putem parametriza traiectoria r(t) In termenii coord. polare

$$\dot{\vec r} = \dot r\,\hat e_r + r\dot\varphi\,\hat e_\theta$$

Lagrangianul in coordonate polare va fi:

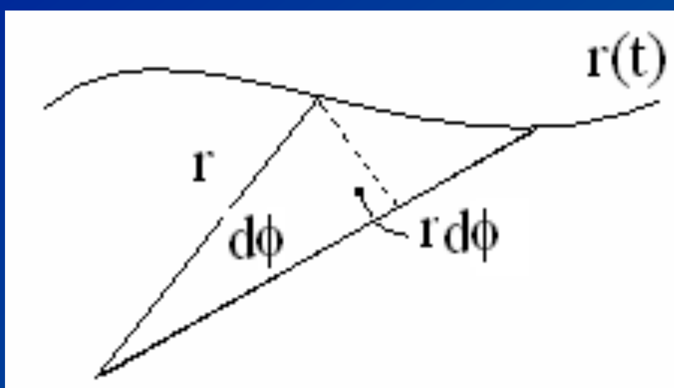
$$L = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\varphi^2\right) - U(r)$$

Observam ca φ este coordonata ciclica, momentul sau conjugat p_φ se conserva

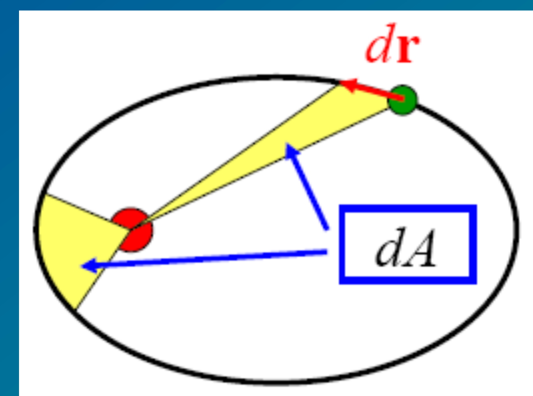
$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = \mu r^2 \dot\varphi \equiv l$$

Marimea momentului unghiular

Introducem notiunea de" viteza areolara"



Legea a II a a lui Kepler: Vectorul de pozitie al unei planete matura arii egale in intervale de timp egale



$$dA = \frac{1}{2} r(r d\varphi) = \frac{1}{2} r^2 d\varphi \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot\varphi = \frac{1}{2}\frac{l}{\mu} = const.$$

viteza areolara

Miscarea planetei este mai rapida cand orbita este mai apropiata de origine

Stabilim ecuatia Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0 \Rightarrow \mu\,\ddot{r}-\mu\,r\,\dot{\varphi}^{2}+\frac{\partial U}{\partial r}=0$$

Forta centrifuga

Forta centrala

$$\dot{\varphi}=\frac{l}{\mu\,r^{2}}$$

$$\mu\,\ddot{r}-\frac{l^{2}}{\mu\,r^{3}}+\frac{\partial U}{\partial r}=0$$

insa

$$\ddot{r}=\frac{d\dot{r}}{dt}=\frac{d\dot{r}}{dr}\frac{dr}{dt}=\frac{d\dot{r}}{dr}\dot{r}$$

$$\mu\frac{d\dot{r}}{dr}\dot{r}=\frac{l^{2}}{\mu\,r^{3}}-\frac{\partial U}{\partial r}$$

$$\int \mu\,\dot{r}\,d\dot{r}=\int\left(\frac{l^{2}}{\mu\,r^{3}}-\frac{\partial U}{\partial r}\right)dr \Rightarrow \frac{1}{2}\mu\,\dot{r}^{2}=-\frac{1}{2}\frac{l^{2}}{\mu\,r^{2}}-U(r)+E$$

$$E=\frac{1}{2}\mu\,\dot{r}^{2}+\frac{1}{2}\frac{l^{2}}{\mu\,r^{2}}+U(r)$$

E= constanta de integrare

$$U_{ef}(r)=\frac{l^{2}}{2\mu\,r^{2}}+U(r)$$

Conservarea energiei

$$E = T + U(r) = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\varphi^2\right) + U(r)$$

Din ecuatia Lagrange

$$\mu\ddot r - \frac{l^2}{\mu r^3} + \frac{\partial U}{\partial r} = 0 \Rightarrow \mu\ddot r = -\frac{d}{dr}\left[U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}\right]$$

Inmultind cu $\dot r$

$$\Rightarrow \mu\dot r\ddot r = -\dot r\frac{d}{dr}\left[U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}\right]$$

deoarece

$$\mu\dot r\ddot r = \frac{d}{dt}\left(\frac{1}{2}\mu\dot r^2\right) \qquad -\dot r\frac{d}{dr} = -\frac{dr}{dt}\frac{d}{dr} = -\frac{d}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\mu\dot r^2 + U(r) + \frac{1}{2}\frac{l^2}{\mu r^2}\right] = 0$$

Astfel

$$\frac{1}{2}\mu\dot r^2 + U(r) + \frac{1}{2}\frac{l^2}{\mu r^2} = const.$$

deoarece

$$\frac{l^2}{2\mu r^2} = \frac{\mu r^2\dot\varphi^2}{2} \Rightarrow$$

$$E = T + U(r) = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\varphi^2\right) + U(r) = const.$$

Miscarea unei particule intr-un potential efectiv

Ecuatia Lagrange devine:

$$\mu \ddot{r} = -\frac{\partial U_{ef}(r)}{\partial r}$$

Energia:

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{ef}(r)$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U_{ef}(r)\right)}$$

$$t = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U_{ef}(r)\right)}} + const \equiv r(t)$$

$$\dot{\varphi} = \frac{l}{\mu r^2} = \frac{d\varphi}{dt}$$

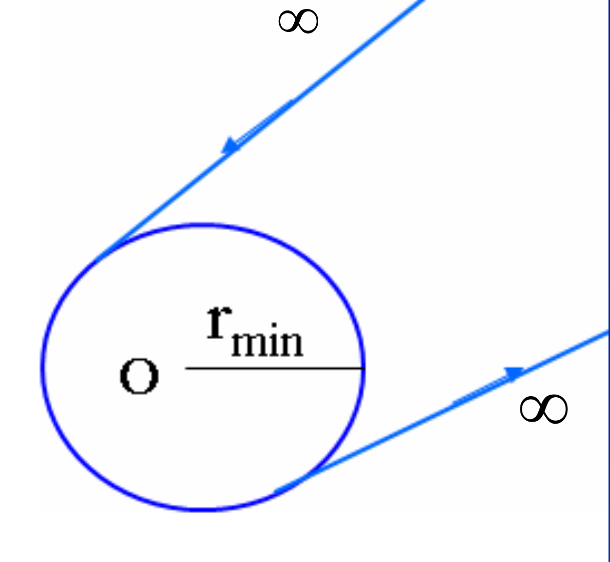
$$\varphi(t) = \int_0^t \frac{l}{\mu\,[r(t)]^2}\,dt + \varphi_0$$

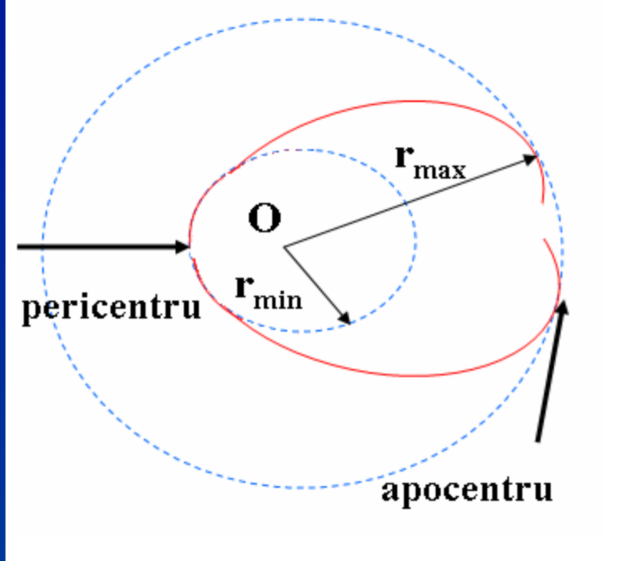
Pentru valori date ale E si l (marimi care se conserva) cautam r(φ) :

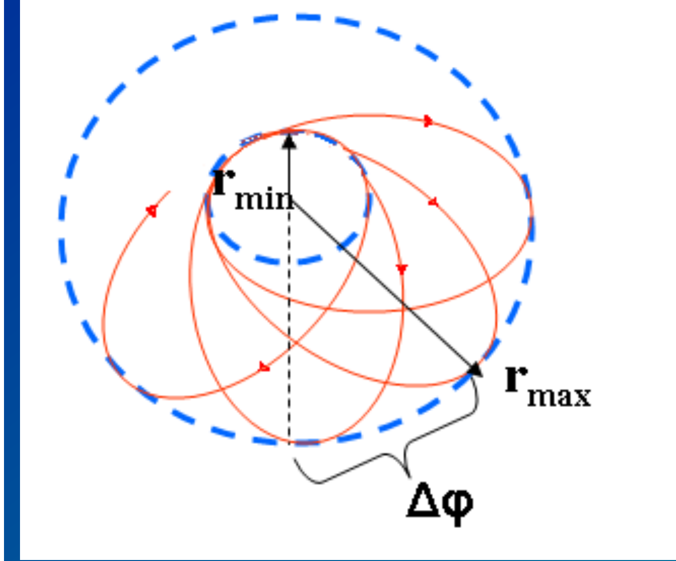
$$\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{dr}{d\varphi}\frac{l}{\mu r^2} = \pm\sqrt{\frac{2}{\mu}\left(E - U_{ef}\right)}$$

$$\varphi = \pm\int_{r_0}^{r} \frac{l}{r^2\sqrt{2\mu(E - U_{ef})}}dr + const \equiv \varphi(r)$$

Ecuatia traiectoriei







$\dot{r}$ are semnificatie fizica daca $E \geq U_{ef}(r)$

- *valorile lui r pentru care* $E = U_{ef}(r)$ *definesc limitele intervalului de valori permise in timpul miscarii*
- *punctul in care* $\dot{r} = 0$ *=punct de intoarcere*
- *daca r_{min} este o radacina pozitiva a ec.* $E = U_{ef}(r)$ *si sunt permise pentru r toate radacinile cuprinse intre (r_{min}, ∞) si daca $r_0 > r_{min}$ misc. particulei este nelimitata*

- *daca ecuatia* $E = U_{ef}(r)$ *are radacini distincte si pozitive, $r_{min} < r_{max}$ si daca in intervalul* $r \in [r_{min}, r_{max}]$ *este verificata inegalitatea* $E \geq U_{ef}(r)$ *atunci miscarea este limitata*
- *Intreaga traiectorie este continuta intr-o coroana circulara*

$$\Delta\varphi = \pm \int_{r_{min}}^{r_{max}} \frac{l}{r^2\sqrt{2\mu(E - U_{ef})}} dr$$

$$n \cdot \Delta\varphi = m \cdot 2\pi; \quad (n, m \geq 1)$$

$$\Delta\varphi = \frac{m}{n} \cdot 2\pi$$

Conditia de"inchidere" a traiectoriei (raza vectoare a punctului, dupa ce a efectuat m rotaii complete, isi va regasi valoarea initiala)

$$U(r) = \begin{cases} -\dfrac{k}{r} \\ kr^2 \end{cases}$$

Ecuatia diferentiala a orbitei

Am gasit forma generala pentru r=r(φ) sau r r(t) si cateva constante E, l etc. si cautam r=r(φ) eliminand parametrul timp, ceea ce inseamna ecuatia orbitei.

$$\mu r^2 \dot{\varphi} = l \Rightarrow \mu r^2 \frac{d\varphi}{dt} = l \Rightarrow \mu r^2 d\varphi = l dt \Rightarrow \frac{d}{dt} = \frac{l}{\mu r^2}\frac{d}{d\varphi}$$

$$\Rightarrow \frac{d^2}{dt^2} = \frac{l}{\mu r^2}\frac{d}{d\varphi}\left(\frac{l}{\mu r^2}\frac{d}{d\varphi}\right)$$

Inlocuind in ecuatia Lagrange

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} = -\frac{\partial U}{\partial r} = f(r)$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{d\varphi}\left(\frac{l}{\mu r^2}\frac{dr}{d\varphi}\right) - \frac{l^2}{\mu r^3} = f(r)$$

Insa $\frac{1}{r^2}\frac{dr}{d\varphi} = -\frac{d}{d\varphi}\left(\frac{1}{r}\right)$ ***si introducand*** $u = \frac{1}{r}$ ***rezulta***

$$\frac{l^2 u^2}{\mu}\left(\frac{d^2 u}{d\varphi^2} + u\right) = -f\left(\frac{1}{u}\right)$$

deoarece $\frac{d}{du} = \frac{dr}{d\varphi}\frac{d}{dr} = -\frac{1}{u^2}\frac{d}{dr}$ ***rezulta***

$$\frac{d^2 u}{d\varphi^2} + u = -\frac{\mu}{l^2}\frac{d}{du}U\left(\frac{1}{u}\right)$$

Ec.diferentiala a orbitei (ecuatia Binet) daca se cunosc f sau U

Pentru un potential oarecare

$$d\varphi = \frac{l\,dr}{\mu r^2\sqrt{\frac{2}{\mu}\left(E - U_{ef}(r)\right)}} = \frac{l\,dr}{\mu r^2\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}$$

$$\varphi = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu U}{l^2} - \frac{1}{r^2}}} + \varphi_0$$

Ecuatie ce da φ ca functie de r si constantele E, l, r_0

Facand schimbarea de variabila $u = \frac{1}{r}$

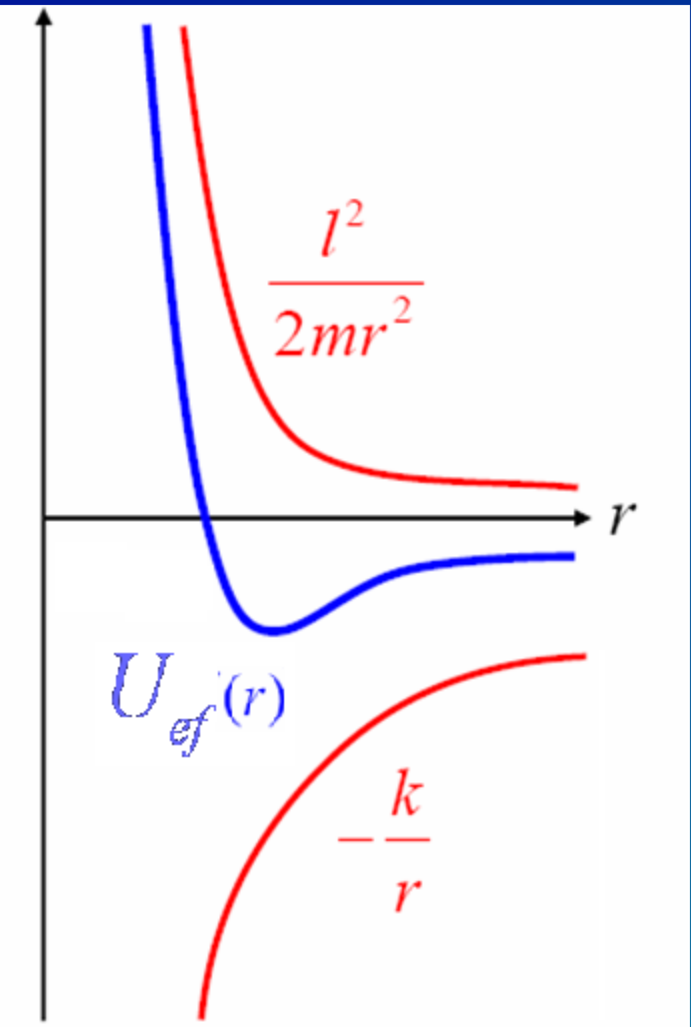
$$\varphi = \varphi_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu U}{l^2} - u^2}}$$

Ecuatia formala a orbitei

Problema lui Kepler

$$f(r) = -\frac{k}{r^2} \Rightarrow U(r) = -\frac{k}{r}$$

$$U_{ef} = -\frac{k}{r} + \frac{l^2}{2mr^2}$$



$$\frac{d^2u}{d\varphi^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right)$$

$$\frac{d^2u}{d\varphi^2} + u = -\frac{m}{l^2}\frac{d}{du} U\left(\frac{1}{u}\right)$$

$$\Rightarrow \frac{d^2u}{d\varphi^2} + u = \frac{mk}{l^2}$$

Facem schimbarea de variabila $y = u - \frac{mk}{l^2}$

$$\Rightarrow \frac{d^2y}{d\varphi^2} + y = 0 \Rightarrow y = C\cos(\varphi - \varphi')$$

C,φ' =const. de integrare

notam $\varepsilon = C\frac{l^2}{mk} \Rightarrow \frac{1}{r} = \frac{mk}{l^2}\left[1 + \varepsilon\cos(\varphi - \varphi')\right]$

$$\varphi = \varphi_0 - \int \frac{du}{\sqrt{\dfrac{2mE}{l^2} - \dfrac{2mU}{l^2} - u^2}}$$

insa $\displaystyle\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{-c}} \arccos\left[-\frac{b+2cx}{\sqrt{b^2-4ac}}\right]$ $\quad a = \dfrac{2mE}{l^2};\ b = \dfrac{2mk}{l^2};\ c = -1$

$$\int d\varphi = -\int \frac{du}{\sqrt{\dfrac{2mE}{l^2} + \dfrac{2mku}{l^2} - u^2}} = -\int \frac{du}{\sqrt{\dfrac{2mE}{l^2} + \dfrac{m^2k^2}{l^4} - \left(\dfrac{mk}{l^2} - u^2\right)^2}} =$$

$$= -\frac{1}{\sqrt{\dfrac{2mE}{l^2} + \dfrac{m^2k^2}{l^4}}} \int \frac{du}{\sqrt{1 - \left(\dfrac{\dfrac{mk}{l^2} - u}{\sqrt{\dfrac{2mE}{l^2} + \dfrac{m^2k^2}{l^4}}}\right)^2}}$$

$$du = \sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^2}} \sin\omega\, d\omega$$

$\cos\omega$

$$= -\int \frac{\sin\omega}{\sin\omega} d\omega = -\omega \quad \Rightarrow \quad \cos\omega = \cos(\varphi - \varphi') = \frac{\frac{mk}{l^2} - u}{\sqrt{\frac{2mE}{l^2} + \frac{m^2k^2}{l^4}}}$$

$$\Rightarrow \quad \boxed{\varphi = \varphi' - \arccos\left[\frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}\right]}$$

deoarece $\boxed{u = \frac{1}{r}}$

$$\Rightarrow \quad \boxed{u = \frac{1}{r} = \frac{mk}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\varphi - \varphi')\right] = \frac{mk}{l^2}\left[1 + \varepsilon \cos(\varphi - \varphi')\right]}$$

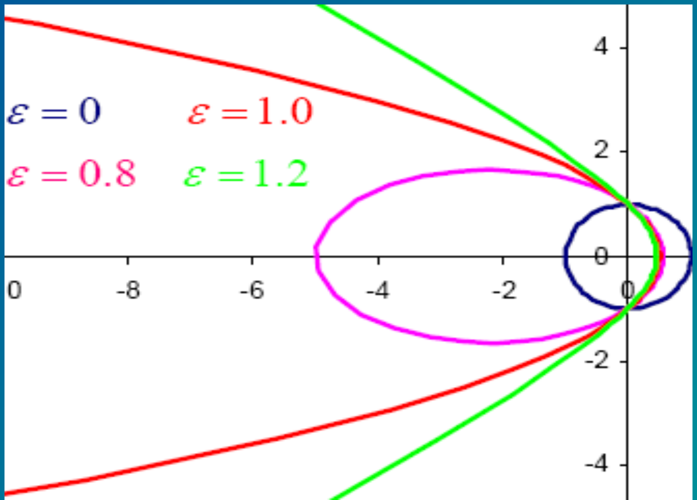
Ecuatia generala a conicei (ε fiind excentricitatea)

$\varepsilon > 1, \quad E > 0: \quad$ *hiperbola*

$\varepsilon = 1, \quad E = 0: \quad$ *parabola*

$\varepsilon < 1, \quad E < 0: \quad$ *elipsa*

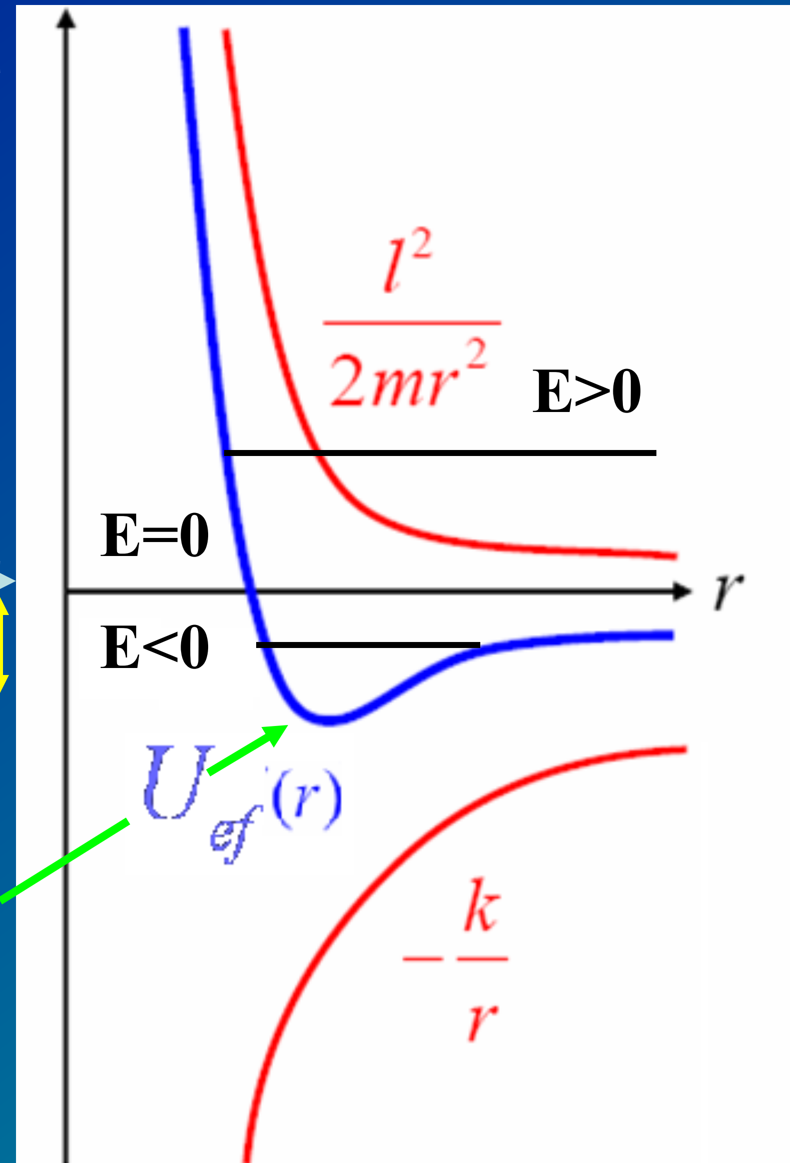
$\varepsilon = 0, \quad E = -\frac{mk^2}{2l^2}: \quad$ *cerc*



Energie si excentricitate *E=0 separa orbitele nemarginite de cele marginite*

Orbite nemarginite
- **Hiperbola**
- **Parabola**

Orbite marginite
- **Elipsa**
- **Cerc**



$$U_{ef}(r_0) = -\frac{k}{r_0} + \frac{l^2}{2mr_0^2} = E$$

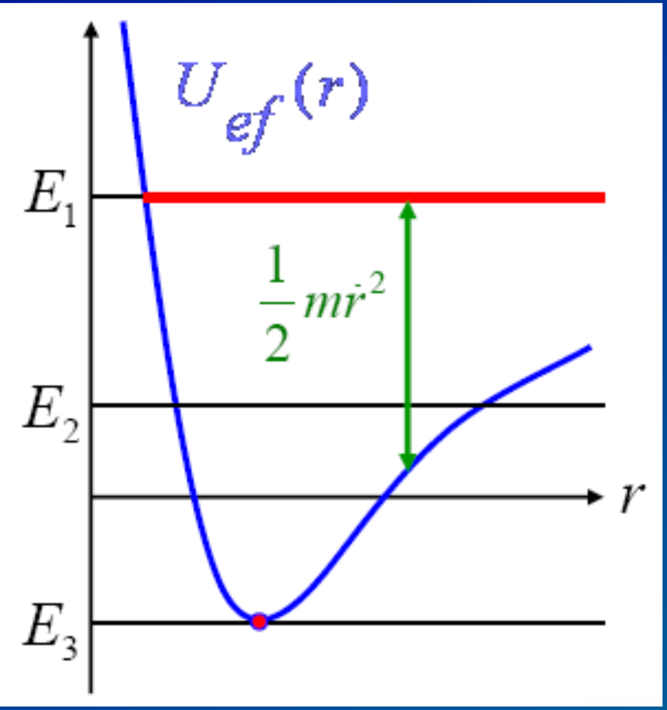
$$\left.\frac{dU_{ef}}{dr}\right|_{r_0} = \frac{k}{r_0^2} - \frac{l^2}{mr_0^3} = 0$$

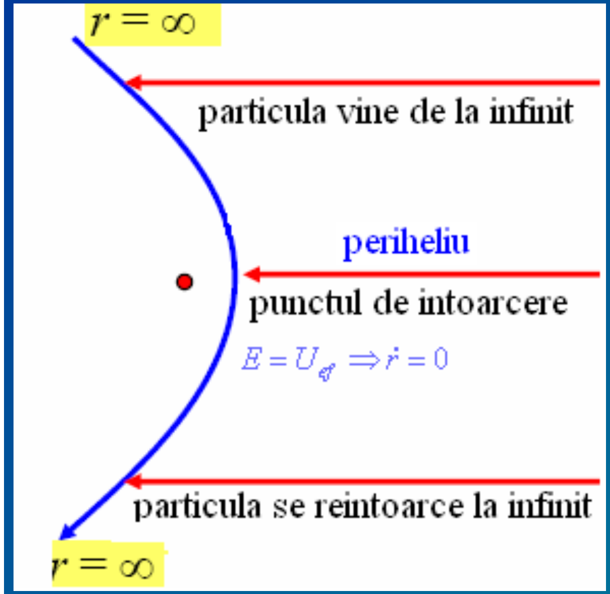
$$E = -\frac{mk^2}{2l^2}$$

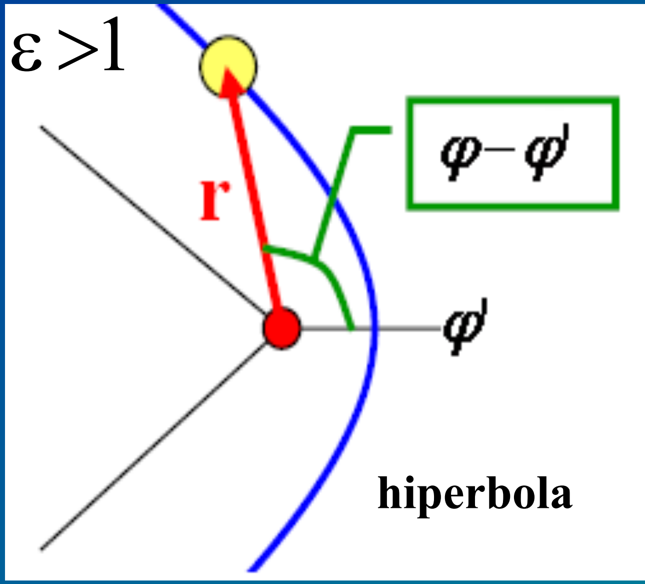
Orbite nemarginite

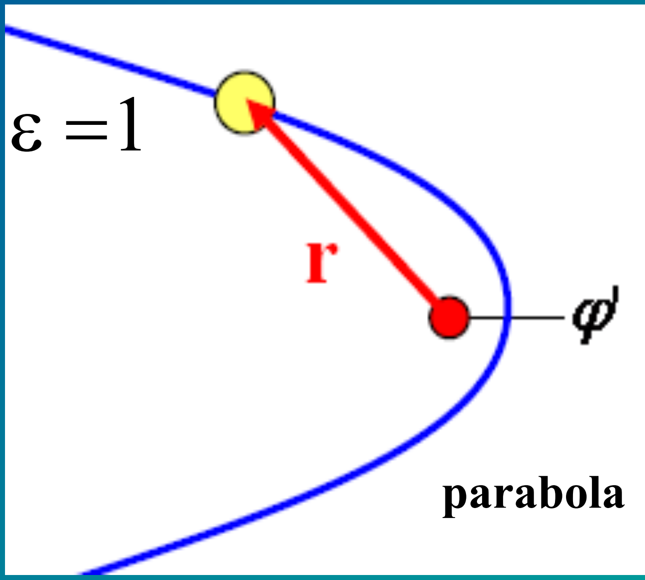
$$E = E_1 \Rightarrow r < r_{min} \qquad E_1 = U_{ef}(r_{min})$$

$$\cos(\varphi - \varphi') > -\frac{1}{\varepsilon}$$ ***limiteaza valoarea lui θ***



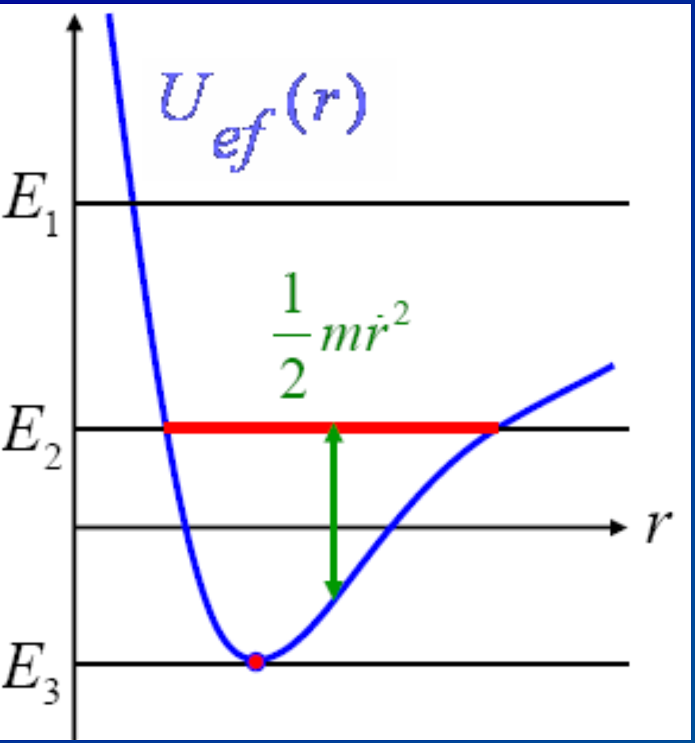


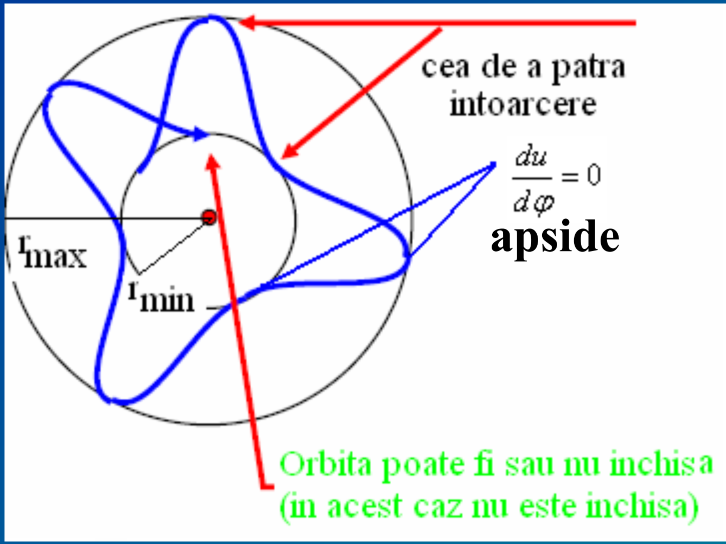




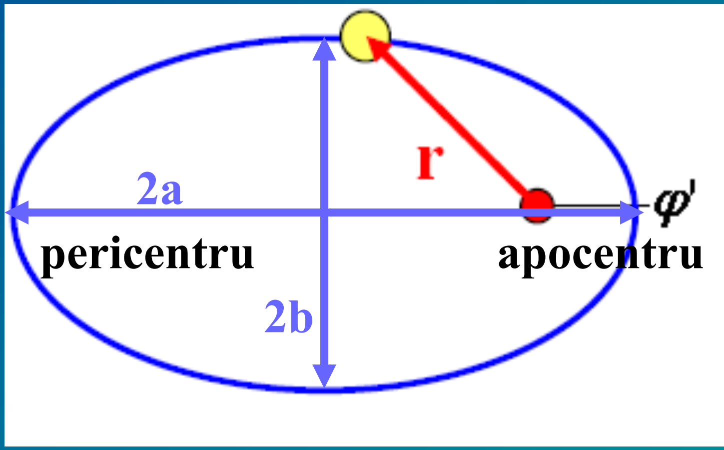
Orbite marginite

$$E = E_2 \Rightarrow r_{\min} < r < r_{\max}$$





$$\frac{1}{r} = \frac{mk}{l^2}(1 \pm \varepsilon)$$



Lungimea axei mari $a = \frac{l^2}{2mk}\left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right) = -\frac{k}{2E}$

Lungimea axei mici $b = a\sqrt{1-\varepsilon^2} = \sqrt{-\frac{l^2}{2mE}}$

Aria orbitei $A = \pi\, a\, b = \pi\sqrt{-\frac{l^2k^2}{8mE^3}}$

Viteza areolara $\dfrac{dA}{dt} = \dfrac{1}{2} r^2 \dot{\varphi}^2 = \dfrac{l}{m}$

Perioada de rotatie $T_{rot} = \dfrac{A}{\left(\dfrac{dA}{dt}\right)} = \pi \sqrt{-\dfrac{mk^2}{2E^3}} = 2\pi \sqrt{\dfrac{m}{k}} a^{\frac{3}{2}}$ **Legea a treia a lui Kepler**

daca $f = -\dfrac{k}{r^2} = -G\dfrac{Mm}{r^2}$

$$T_{rot} = 2\pi \sqrt{\frac{\mu}{k}} a^{\frac{3}{2}} = \beta\, a^{\frac{3}{2}}; \quad \beta = 2\pi \sqrt{\frac{1}{G(m+M)}}$$

este acelasi pentru toate planetele daca M>>m